

# Quasi-exactly solvable quasinormal modes

Choon-Lin Ho and Hing-Tong Cho

Department of Physics, Tamkang University, Tamsui 251, Taiwan, Republic of China

We consider quasinormal modes with complex energies from the point of view of the theory of quasi-exactly solvable (QES) models. We demonstrate that it is possible to find new potentials which admit exactly solvable or QES quasinormal modes by suitable complexification of parameters defining the QES potentials. Particularly, we obtain one QES and four exactly solvable potentials out of the five one-dimensional QES systems based on the  $sl(2)$  algebra.

**Introduction.**— Quasinormal modes (QNM) arise as perturbations of stellar or black hole spacetimes [1]. They are solutions of the perturbation equations that are outgoing to spatial infinity and the event horizon. Generally, these conditions lead to a set of discrete complex eigenfrequencies, with the real part representing the actual frequency of oscillation and the imaginary part representing the damping. QNM carry information of black holes and neutron stars, and thus are of importance to gravitational-wave astronomy. In fact, these oscillations, produced mainly during the formation phase of the compact stellar objects, can be strong enough to be detected by several large gravitational wave detectors under construction. Recently, QNM of particles with different spins in black hole spacetimes have also received much attention.

Owing to the intrinsic complexity in solving the perturbation equations in general relativity with the appropriate boundary conditions, one has to resort to various approximation methods, eg., the WKB method, the phase-integral method etc., in obtaining QNM solutions. It is therefore helpful that one can get some insights from exact solutions in simple models, such as the inverted harmonic oscillator and the Pöschl-Teller potential. Unfortunately, the number of exactly solvable models is rather limited.

Recently, in non-relativistic quantum mechanics a new class of potentials which are intermediate to exactly solvable ones and non-solvable ones has been found. These are called quasi-exactly solvable (QES) problems for which it is possible to determine analytically a part of the spectrum but not the whole spectrum [2, 3, 4, 5]. The discovery of this class of spectral problems has greatly enlarged the number of physical systems which we can study analytically. In the last few year, QES theory has also been extended to the Pauli and Dirac equations.

In [6] we have considered solutions of QNM based on the Lie-algebraic approach of QES theory. There we demonstrate that, by suitable complexification of some parameters of the generators of the  $sl(2)$  algebra while keeping the Hamiltonian Hermitian, we can indeed obtain potentials admitting exact or quasi-exact QNMs. Such consideration has not been attempted before in studies of QES theory. Our work represents a direct opposite of the work in [7], where QES *real energies* were obtained from a *non-Hermitian* PT-symmetric Hamiltonian.

**QES Theory.**— Let us briefly review the essence of the Lie-algebraic approach [2, 3, 5] to QES models. Consider a Schrödinger equation  $H\psi = E\psi$  with Hamiltonian  $H = -d_x^2 + V(x)$  ( $d_x \equiv d/dx$ ) and wave function  $\psi(x)$ . Here  $x$  belongs either to the interval  $(-\infty, \infty)$  or  $[0, \infty)$ . Now suppose we make an “imaginary gauge transformation” on the function  $\psi$ :  $\psi(x) = \chi(x)e^{-g(x)}$ , where  $g(x)$  is called the gauge function. For physical systems which we are interested in, the phase factor  $\exp(-g(x))$  is responsible for the asymptotic behaviors of the wave function so as to ensure normalizability. The function  $\chi(x)$  satisfies a Schrödinger equation with a gauge transformed Hamiltonian  $H_g = e^g H e^{-g}$ . Suppose  $H_g$  can be written as a quadratic combination of the generators  $J^a$  of some Lie algebra with a finite dimensional representation. Within this finite dimensional Hilbert space the Hamiltonian  $H_g$  can be diagonalized, and therefore a finite number of eigenstates are solvable. Then the system described by  $H$  is QES. For one-dimensional QES systems the most general Lie algebra is  $sl(2)$ , and  $H_g$  can be expressed as

$$H_g = \sum C_{ab} J^a J^b + \sum C_a J^a + \text{real constant} , \quad (1)$$

where  $C_{ab}$ ,  $C_a$  are taken to be *real constants* in [3, 5]. The generators  $J^a$  of the  $sl(2)$  Lie algebra take the differential forms:  $J^+ = z^2 d_z - n z$ ,  $J^0 = z d_z - n/2$ ,  $J^- = d_z$  ( $n = 0, 1, 2, \dots$ ). The variables  $x$  and  $z$  are related by some function to be described later.  $n$  is the degree of the eigenfunctions  $\chi$ , which are polynomials in a  $(n+1)$ -dimensional Hilbert space with the basis  $\{1, z, z^2, \dots, z^n\}$ .

Substituting the differential forms of  $J^a$  into Eq. (1), one sees that every QES operator  $H_g$  can be written in the canonical form :  $H_g = -P_4(z)d_z^2 + P_3(z)d_z + P_2(z)$ , where  $P_k(z)$  are  $k$ -th degree polynomial in  $z$  with real coefficients related to the constants  $C_{ab}$  and  $C_a$ . The relation between  $H_g$  and the standard Schrödinger operator  $H$  fixes the required form of the gauge function  $g$  and the transformation between the variable  $x$  and  $z$ . Particularly,  $x = \int^z dy / \sqrt{P_4(y)}$ . Analysis on the inequivalent forms of real quartic polynomials  $P_4$  thus give a classification of all  $sl(2)$ -based QES Hamiltonians [3, 5]. If one imposes the requirement of non-periodic potentials, then there are only five inequivalent classes, which are called case 1 to 5 in [5].

Our main observation is this. If some of the coefficients in  $P_k(z)$  are allowed to be complex while keeping  $V(x)$  real, then all the five cases classified in [5] can indeed support QES/exact quasinormal modes. We shall illustrate this using one of the cases below.

**QES QNM.**— We consider Case 3 in [5], which corresponds to Class I potential in Turbiner's scheme [3]. There are two subclasses in this case, namely, Case (3a) and (3b). We shall present the analysis of QNM potential for case (3a). The other case turns out to give the same potential with a suitable choice of the parameters. The potential in case (3a) has the form (in this paper we adopt the notation of [5]):

$$V(x) = Ae^{2\sqrt{\nu}x} + Be^{\sqrt{\nu}x} + Ce^{-\sqrt{\nu}x} + De^{-2\sqrt{\nu}x}, \quad (2)$$

where  $x \in (-\infty, \infty)$  and  $\nu$  is a positive scale factor. Note that  $V(x)$  is defined up to a real constant, which we omit for simplicity, as it merely shifts the real part of the energy. This remark also applies to the other cases.  $V(x)$  in Eq. (2) reduces to the exactly solvable Morse potentials when  $A = B = 0$ , or  $C = D = 0$ . This potential is QES when the coefficients are related by

$$\begin{aligned} A &= \frac{\hat{b}^2}{4\nu}, \quad B = \frac{\hat{c} + (n+1)\nu}{2\nu} \hat{b}, \\ C &= \frac{\hat{c} - (n+1)\nu}{2\nu} \hat{d}, \quad D = \frac{\hat{d}^2}{4\nu}, \\ n &= 0, 1, 2, \dots \end{aligned} \quad (3)$$

Here  $\hat{b}$ ,  $\hat{c}$ ,  $\hat{d}$  are arbitrary real constants. For each integer  $n \geq 0$ , there are  $n+1$  exactly solvable eigenfunctions in the  $(n+1)$ -dimensional QES subspace:

$$\psi_n(x) = \exp \left[ \frac{\hat{b}}{2\nu} e^{\sqrt{\nu}x} + \frac{\hat{c} - n\nu}{2\sqrt{\nu}} x - \frac{\hat{d}}{2\nu} e^{-\sqrt{\nu}x} \right] \chi_n(e^{\sqrt{\nu}x}). \quad (4)$$

Here  $\chi_n(z)$  is a polynomial of degree  $n$  in  $z = \exp(\sqrt{\nu}x)$ . To guarantee normalizability of the eigenfunctions, the real constants  $\hat{b}$ ,  $\hat{c}$ ,  $\hat{d}$ ,  $\nu$  and  $n$  must satisfy certain relations [5].

We want to see if we can get QNM solutions if we allow some parameters to be complex, while still keeping the potential  $V(x)$  real. This latter requirement restricts the possible values of the parameters, and hence the forms of QES potential admitting quasinormal modes. For the case at hand, we find that one possible choice of values of  $\hat{b}$ ,  $\hat{c}$  and  $\hat{d}$  is:

$$\hat{b} = ib, \quad \hat{c} = -(n+1)\nu, \quad \hat{d} = d, \quad b, d : \text{real constants}. \quad (5)$$

The potential Eq. (2) becomes

$$V_n(x) = -\frac{b^2}{4\nu} e^{2\sqrt{\nu}x} - (n+1)d e^{-\sqrt{\nu}x} + \frac{d^2}{4\nu} e^{-2\sqrt{\nu}x}, \quad (6)$$

and the wave function Eq. (4) becomes

$$\begin{aligned} \psi_n(x) &= \exp \left[ \frac{ib}{2\nu} e^{\sqrt{\nu}x} - \left( n + \frac{1}{2} \right) \sqrt{\nu}x - \frac{d}{2\nu} e^{-\sqrt{\nu}x} \right] \\ &\quad \times \chi_n(e^{\sqrt{\nu}x}). \end{aligned} \quad (7)$$

$V(x)$  approaches  $\mp\infty$  as  $x \rightarrow \pm\infty$  respectively: it is unbounded from below on the right. For small positive  $d$  and sufficiently large  $n$ ,  $V(x)$  can have a local minimum and a local maximum. In this case, the well gets shallower as  $d$  increases at fixed value of  $n$ , or as  $n$  decreases at fixed  $d$ . We emphasize here that for different value of  $n$ , each  $V(x)$  represents a different QES potential admitting  $n+1$  QES solutions. Since  $V(x) \rightarrow \infty$  as  $x \rightarrow -\infty$ , the wave function must vanish in this limit. This means  $d > 0$  from Eq. (7). For the outgoing boundary condition, we must take  $b > 0$ . Before we go on, we note here that there is another possible choice of the parameters, namely,

$$\hat{b} = -b, \quad \hat{c} = (n+1)\nu, \quad \hat{d} = id, \quad b, d : \text{real constants}. \quad (8)$$

However, this choice leads to a potential related to Eq. (6) by the reflection  $x \mapsto -x$ . Hence, we will only discuss the potential in Eq. (6) here.

To see that the wave functions  $\psi_n(x)$  do represent quasinormal modes, we determine the corresponding energy  $E_n$ . This is easily done by solving the eigenvalue problem of the polynomial part  $\chi_n(z)$  of the wave function. From the Schrödinger equation we find that for  $n = 0$ , the energy is  $E_0 = -\nu/4 - ibd/2\nu$ . This clearly shows that the only QES solution when  $n = 0$  is a QNM with an energy having a negative imaginary part (recall that  $b, d, \nu > 0$ ). For  $n = 1$ , we have two QES solutions. Their energies are  $E_1 = -5\nu/4 - ibd/2\nu \pm \sqrt{\nu^2 - ibd}$ . Again we have two QNM modes. One can proceed accordingly to obtain  $n + 1$  QNM modes with higher values of  $n$ . However, for large  $n$ , computation becomes tedious, and one has to resort to numerical means.

One can do the same for the other four cases listed in [5]. It is found that these cases admits exact QNM solutions.

**Summary.**— To summarize, we have demonstrated that it is possible to extend the usual QES theory to accommodate QNM solutions, by complexifying certain parameters defining the QES potentials. We found that the five  $sl(2)$ -based QES systems listed in [5] can be so extended. While one of these cases admits QES QNM, the other four cases give exact QNM solutions. It is hoped that our work would motivate the search of many more exact/quasi-exact systems of QNM in QES theories based on higher Lie algebras, and in higher dimensions.

This work was supported in part by the National Science Council of the Republic of China under the Grants NSC 94-2112-M-032-009 (H.T.C.) and NSC 94-2112-M-032-007 (C.L.H.).

- 
- [1] For a good review, see eg.: K.D. Kokkotas and B.G. Schmidt, Living Rev. Rel. **2**, 2 (1999); S. Chandrasekhar, *The mathematical theory of black holes* (Clarendon, Oxford, 1983).
  - [2] A. Turbinder and A.G. Ushveridze, Phys. Lett. A **126**, 181 (1987).
  - [3] A.V. Turbinder, Comm. Math. Phys. **118**, 467 (1988).
  - [4] A.G. Ushveridze, Sov. Phys.-Lebedev Inst. Rep. **2**, 50, 54 (1988); *Quasi-exactly solvable models in quantum mechanics* (IOP, Bristol, 1994).
  - [5] A. González, N. Kamran and P.J. Olver, Comm. Math. Phys. **153**, 117 (1993).
  - [6] H.-T. Cho and C.-L. Ho, J.Phys. A **40**, 1325 (2007).
  - [7] C.M. Bender and S. Boettcher, J. Phys. A **31**, L273 (1998).